# Section 15.3

## Triple Integrals



Introduction

- Applications and Physical Meaning
- Fubini's Theorem and Iterated Integrals on Rectangular Prisms

Simple Solids

- Examples, Setting up the Integral
- Example, When a Solid is not $x$-Simple
- Example, Change of Order of Integration

# 1 Introduction

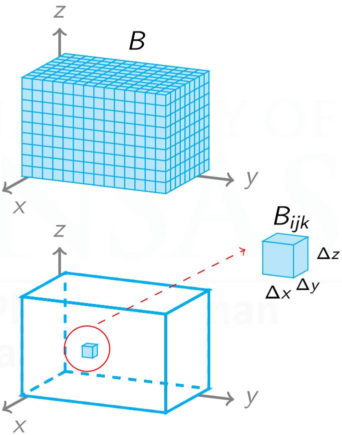
# Triple Integrals

Triple integrals are analogous to double and single integrals.

Though difficult to visualize, the triple integral

$$\iiint_B f(x, y, z)\, dV$$

calculates the net amount of 4-dimensional space contained within the function $f$ above the solid $B$.




$$\iiint_B f(x, y, z)\, dV = \lim_{(L,M,N)\to(\infty,\infty,\infty)} \sum_{i=1}^{L} \sum_{j=1}^{M} \sum_{k=1}^{N} f(x_i, y_j, z_k)\, \Delta x\, \Delta y\, \Delta z$$

# Triple Integrals

Some concrete applications of triple integrals:

- Volume of a solid region $W = \iiint_W 1\, dV$

- Average value of $f$ on $W = \dfrac{\iiint_W f(x,y,z)\, dV}{\text{volume}(W)}$

- Total amount of "stuff" in $W = \iiint_W \delta(x,y,z)\, dV$

  where $\delta(x,y,z) =$ density at point $(x,y,z)$

These applications occur frequently in physics and engineering problems.

## Fubini's Theorem for Triple Integrals

If $f$ is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

The iterated integral may be evaluated in any of the 6 possible orders.

**Example 1:** Let $B = [0, 1] \times [-1, 2] \times [0, 3]$.

$$\begin{aligned} \iiint_B xyz^2\, dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz \\ &= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz \\ &= \int_0^3 \frac{3z^2}{4}\, dz = \frac{27}{4} \end{aligned}$$

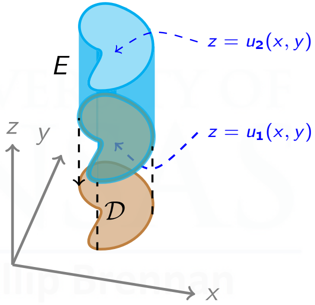
# 2 Simple Solids

# Triple Integrals on Simple Solids

A solid $E$ is called **z-simple** if the **z** coordinate is bounded by two continuous functions of $x$ and $y$.

$$D = \{(x,y) \mid (x,y,z) \in E\}$$

$$E = \left\{(x,y,z) \;\middle|\; \begin{array}{l} (x,y) \in D, \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$$




$$\iiint_E f(x,y,z)\,dV = \iint_D \left( \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

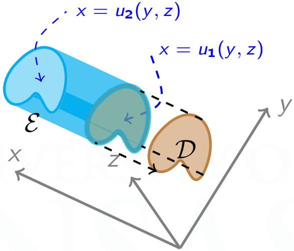
Calculating the inner integral produces a function of $x$ and $y$.
The triple integral then becomes a double integral.

$E$ is a x-**simple solid** if the **x** coordinate is bounded by two continuous functions of $y, z$.

$D$ is the projection of $E$ onto the $yz$-plane.

$$D = \{(y,z) \mid (x,y,z) \in E\}$$

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} (y,z) \in D, \\ u_1(y,z) \le x \le u_2(y,z) \end{array} \right\}$$

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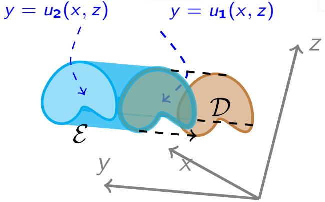



$E$ is a y-**simple solid** if the **y** coordinate is bounded by two continuous functions of $x, z$.

$D$ is the projection of $E$ onto the $xz$-plane.

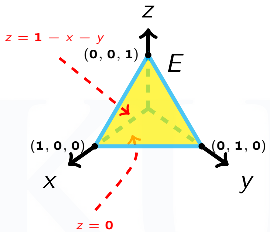
$$D = \{(x,z) \mid (x,y,z) \in E\}$$

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} (x,z) \in D, \\ u_1(x,z) \le y \le u_2(x,z) \end{array} \right\}$$

## Steps for Integration Set up

1. Draw the the domain in $\mathbb{R}^3$ (a solid).
2. Decide if it is $x$, $y$ or $z$-simple; solve the lower and the upper surface for $x$, $y$ or $z$ in terms of $(y, z)$, $(x, z)$ or $(x, y)$ respectively. Set up the most inner integral using those surfaces.
3. If $x$, $y$ or $z$-simple, find the projection of the domain in $yz$, $xz$ or $xy$ planes, respectively. Set up the two outer integral as a double integral over this region.

**Example 2:** Find the volume of $E$, where $E$ is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

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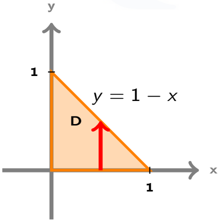



**Projection of the domain on *xy*-plane**

Solution: The solid can be viewed as an x-, y-, or z-simple solid.

Points $(x, y, z)$ in $E$ are such that

$$0 \leq z \leq 1 - x - y$$

where $(x, y) \in D$ which is vertically simple:

$$0 \leq y \leq 1 - x \qquad 0 \leq x \leq 1$$

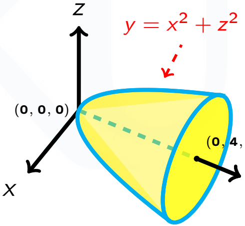
$$\begin{aligned} \text{Volume} &= \overbrace{\int_0^1 \int_0^{1-x}}^{\text{Double Integral}} \int_0^{1-x-y} 1 \, dz \, dy \, dx \\ &= \int_0^1 \int_0^{1-x} 1 - x - y \, dy \, dx \\ &= \int_0^1 \left( y - xy - \frac{y^2}{2} \right) \Big|_0^{1-x} dx \\ &= \int_0^1 (1 - x) - x(1 - x) - \frac{(1 - x)^2}{2} \, dx = \frac{1}{6} \end{aligned}$$

**Example 3:** Let $E$ be the solid bounded by the elliptical paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Express

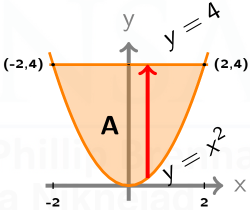
$$\iiint_E f(x, y, z)\, dV$$

as an iterated integral by viewing $E$ as a z-simple solid.

**Domain:**




**Projection of the Domain:**




- $A$ is the (vertically simple) projection of $E$ onto the $xy$-plane.
- If $(x, y, z)$ in $E$, then $-\sqrt{y - x^2} \leq z \leq \sqrt{y - x^2}$.

Solution:

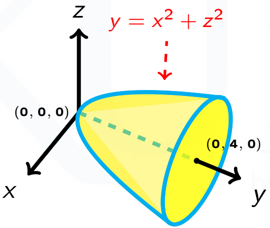
$$\int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} f(x, y, z)\, dz\, dy\, dx$$

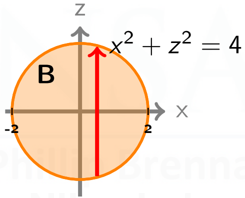
Link

**Example 4:** Let $E$ be the solid bounded by the elliptical paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Express

$$\iiint_E f(x, y, z)\, dV$$

as an iterated integral by viewing $E$ as a y-simple solid.





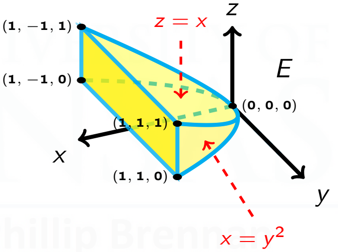

- $B$ is the (vertically simple) projection of $E$ onto the $xz$-plane.
- If $(x, y, z)$ in $E$, then $x^2 + z^2 \leq y \leq 4$.

<u>Solution:</u> $$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} f(x, y, z)\, dy\, dz\, dx$$ Video

**Example 5:** Find the volume of the solid $E$ bounded by the surfaces

$$x = y^2, \qquad x = z, \qquad z = 0, \qquad x = 1.$$

**Domain:**




Solution: It is easiest to treat this solid as z-simple. It is defined by the inequalities

$$-1 \leq y \leq 1, \qquad y^2 \leq x \leq 1, \qquad 0 \leq z \leq x.$$

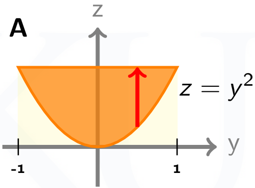
$$\text{Volume} = \iiint_E dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x dz\, dx\, dy = \int_{-1}^{1} \int_{y^2}^{1} x\, dx\, dy$$

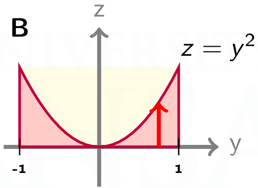
$$= \int_{-1}^{1} \frac{1 - y^4}{2}\, dy = \frac{4}{5}$$

▸ Link ▸ Video

# Example 5 is not $x$-simple

View the $yz$-projection: Link Two surfaces $x = z$ and $x = y^2$ create a piecewise defined upper bound. Eliminate $x$ between the two equations to find the projection of their intersection; $z = y^2$.




(A) The volume of the solid above $A$:

$$\int_{-1}^{1} \int_{y^2}^{1} \int_{z}^{1} 1 \, dx \, dz \, dy$$

(B) The volume of the solid above $B$:

$$\int_{-1}^{1} \int_{0}^{y^2} \int_{y^2}^{1} 1 \, dx \, dz \, dy$$

The volume of $E$ is the sum of the two integrals.

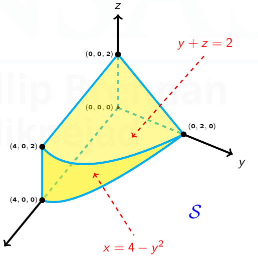
**Example 6:** Rewrite the triple iterated integral

$$\int_0^2 \int_0^{4-y^2} \int_0^{2-y} f(x,y,z)\, dz\, dx\, dy$$

in the order $dx\, dy\, dz$.

Solution: As always, start by sketching the region. The order of the limits of integration can be changed either algebraically or geometrically:

$$\begin{aligned} 0 &\le y \le 2 \\ 0 &\le x \le 4-y^2 \\ 0 &\le z \le 2-y \end{aligned} \quad \longrightarrow \quad \begin{aligned} 0 &\le z \le 2 \\ 0 &\le y \le 2-z \\ 0 &\le x \le 4-y^2 \end{aligned}$$




Answer: $\displaystyle\int_0^2 \int_0^{2-z} \int_0^{4-y^2} f(x,y,z)\, dx\, dy\, dz$

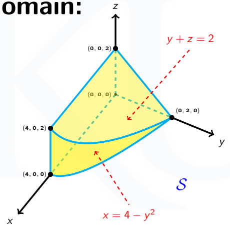
Link

**More on Example 6:** Represent the volume of the solid $\mathcal{S}$ three different ways as an iterated triple integral, where $\mathcal{S}$ is bounded by the surfaces

$$y + z = 2 \qquad x = 4 - y^2 \qquad x = 0 \qquad y = 0 \qquad z = 0$$

Solution:

**Domain:**

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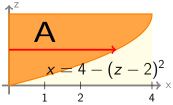
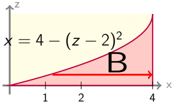
z-Simple: $\displaystyle\int_0^2 \int_0^{4-y^2} \int_0^{2-y} f \, dz \, dx \, dy$

x-Simple: $\displaystyle\int_0^2 \int_0^{2-z} \int_0^{4-y^2} f \, dx \, dy \, dz$

**Not y-Simple!**

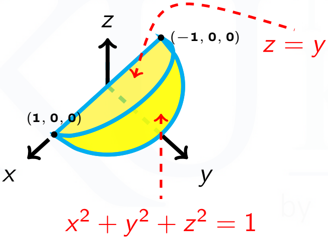
$$\underbrace{\left(\int_0^2 \int_0^{4-(z-2)^2}}_{A} \int_0^{2-z} f \, dy \, dx \, dz\right) + \underbrace{\left(\int_0^2 \int_{4-(z-2)^2}^{4}}_{B} \int_0^{\sqrt{4-x}} f \, dy \, dx \, dz\right)$$

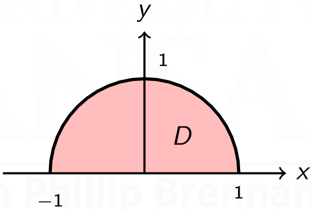


# Example 7, Non-simple Region Setup

**Example 7:** Let $W$ be the solid region inside the unit sphere $x^2 + y^2 + z^2 = 1$, above the plane $z = 0$, and below the plane $z = y$. Set up a triple integral over $W$ as an iterated integral.



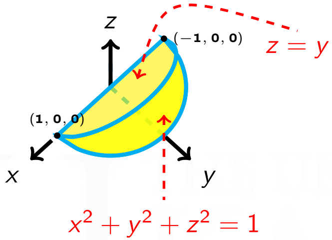



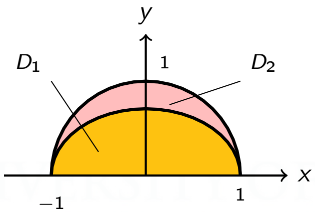



Solution: The first step is to draw a picture of $W$.
The projection of $W$ into the $x, y$-plane is a half-disk:

$$D = \{(x, y) \mid y \geq 0,\ x^2 + y^2 \leq 1\}.$$

**Example 7 (continued)**




The region $W$ is not $z$-simple:

$$W = \{(x, y, z) \mid (x, y) \in D,\ 0 \leq z \leq u(x, y)\}.$$

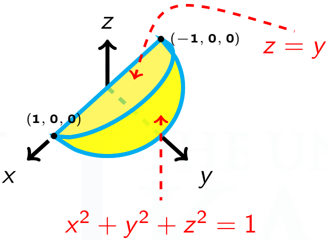
The upper bound $u(x, y)$ has to be defined piecewise.

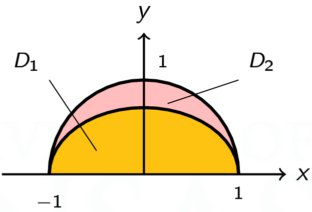
The intersection of the plane $z = y$ and the sphere $x^2 + y^2 + z^2 = 1$ satisfies $x^2 + 2y^2 = 1$. This is the boundary between the two pieces:

$$u(x, y) = \begin{cases} y & \text{if } x^2 + 2y^2 \leq 1, \\ \sqrt{1 - x^2 - y^2} & \text{if } x^2 + 2y^2 \geq 1. \end{cases}$$

Link Video

**Example 7 (continued)**



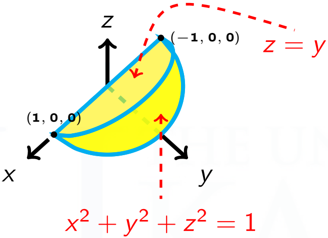



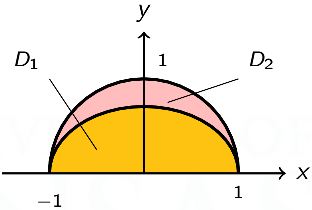

Split up $D$ into two subregions $D_1, D_2$. Each is $z$-simple:

$$D_1 = \left\{(x, y) : \ -1 \le x \le 1, \ 0 \le y \le \sqrt{\frac{1 - x^2}{2}}\right\},$$

$$D_2 = \left\{(x, y) : \ -1 \le x \le 1, \ \sqrt{\frac{1 - x^2}{2}} \le y \le \sqrt{1 - x^2}\right\}.$$

**Example 7 (continued)**



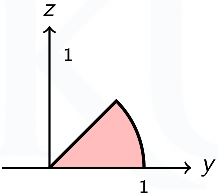

Putting the entire information together:

$$\iiint_W f\,dV = \iint_{D_1}\left(\int_0^{y} f\,dz\right)\,dA + \iint_{D_2}\left(\int_0^{\sqrt{1-x^2-y^2}} f\,dz\right)\,dA$$

$$= \boxed{\int_{-1}^{1}\int_0^{\sqrt{\frac{1-x^2}{2}}}\int_0^{y} f\,dz\,dy\,dx \;+\; \int_{-1}^{1}\int_{\sqrt{\frac{1-x^2}{2}}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} f\,dz\,dy\,dx.}$$

# Example 7, Simple Regions Setup

This solid is both $x$-simple and $y$-simple:




$x$-simple setup:

$$\iiint_W f\,dV = \int_0^1 \int_y^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f\,dx\,dz\,dy$$

$y$-simple Setup:

$$\iiint_W f\,dV \quad = \int_{-1}^1 \int_y^{\sqrt{1-2z^2}} \int_z^{\sqrt{1-x^2-z^2}} f\,dy\,dz\,dx$$